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On Zagier's adele

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Abstract

Purpose: Don Zagier suggested a natural construction, which associates a real number and p-adic numbers for all primes p to the cusp form $g = \Delta$ of weight 12. He claimed that these quantities constitute a rational adele. In this paper we prove this statement, and, more importantly, a similar statement when g is a weight 2 primitive form with rational integer Fourier coefficients.

Methods: While a simple modular argument suffices for the proof of Zagier's original claim, consideration of the case when g is of weight 2 involves Hodge decomposition for the formal group law of the rational elliptic curve associated with g.

Results and Conclusions: While in the weight 12 setting considered by Zagier the claim under consideration depends on a specific choice of a mock modular form which is good for g, in the case when g is of weight 2, the statement has a global nature, and depends on the fact that the classical addition law for the Weierstrass ζ-function is defined over $\mathbb{Z}[1/6]$.

MSC: 11F37; 14H52; 14L05

Background

Let

$$g = \sum_{n\geq 1} b(n)q^n \in S_k(N),$$

with $q = \exp(2\pi i\tau)$ and $\Im(\tau) > 0$ be a primitive form of conductor N (i.e., a new normalized cusp Hecke eigenform on $\Gamma_0(N)$, cf. [1], Section 4.6) of even integer weight k. Assume that all Fourier coefficients $b(n) \in \mathbb{Z}$ are rational integers.

We denote by $\mathcal{E}_g$ the Eichler integral

$$\mathcal{E}_g(\tau) = \sum_{n\geq 1} n^{1-k} b(n)q^n$$

associated with g.

Let M be a weak harmonic Maass form which is good for g in the sense of [2] (see Section 'Proof of Theorem 1' below for the relevant definitions), and let

$$M = M^+ + M^-$$

be the canonical decomposition of M into its non-holomorphic part M^- and a mock modular form M^+. Although the mock modular form M^+ does not typically have rational Fourier coefficients, Theorem 1.1 in [3], implies the existence of $\alpha \in \mathbb{R}$ such that

$$\mathcal{M}_\alpha := M^+ - \alpha\mathcal{E}_g \in \mathbb{Q}((q)).$$

Here and throughout, $\phi \in K((q))$ means that $q^t\phi \in K[\![q]\!]$ for some positive integer t.
For a prime p, the operators $U = U_p$ and $V = V_p$ act on $K((q))$ by

$$\left(\sum_{n\gg-\infty} c(n)q^n\right)\Bigg| U = \sum_{n\gg-\infty} c(pn)q^n, \qquad \left(\sum_{n\gg-\infty} c(n)q^n\right)\Bigg| V = \sum_{n\gg-\infty} c(n)q^{pn},$$

and we will suppress the index p when that does not lead to confusion.

The coefficients of the series $\mathcal{M}_\alpha$ typically have unbounded denominators. More specifically (see [3,4] and Proposition 5 below), for almost all primes p, there exist $\lambda_p, \mu_p \in \mathbb{Q}_p$ with $\mu_p = 0$ if $p \nmid b(p)$ such that the powers of p in the denominators of the coefficients of the series

$$\mathcal{M}_\alpha - \lambda_p\mathcal{E}_g - \mu_p\mathcal{E}_g|V_p \in \mathbb{Q}_p((q))$$

are bounded. The mock modular form M^+ is defined modulo an addition of a weakly holomorphic modular form $h \in M^!_{2-k}(N)$ which is bounded at all cusps except infinity and has rational Fourier coefficients at infinity. However, since the Fourier coefficients of h must have bounded denominators, the choice of M^+ does not affect the quantities $\lambda_p, \mu_p \in \mathbb{Q}_p$. It was Zagier (unpublished) who first considered the quantities λ_p in the case when $k = 12$ and $g = \Delta \in S_{12}(1)$: he claimed that there is an 'optimal' p-adic multiple of $\mathcal{E}_g$ to correct $\mathcal{M}_\alpha$. In this case, Zagier observed the following phenomenon (see Proposition 7 in Section 'Proof of Theorem 1' for a proof).

Proposition 1. *If $g = \Delta \in S_{12}(1)$, then for all but possibly finitely many primes p, we have that $\lambda_p \in \mathbb{Z}_p$.*

Proposition 1 means that the sequence $(-\alpha, (\lambda_p))$ is an adele of $\mathbb{Q}$. Note that there is a considerable freedom in the choice of α: for any $r \in \mathbb{Q}$, the quantity $\alpha + r$ will do the same job. Furthermore, if one picks $\alpha + r$ instead of α, then λ_p becomes $\lambda_p - r$ for all p shifting the adele by the principal adele (r). Equivalently, there exists a choice of α such that $\lambda_p \in \mathbb{Z}_p$ for all primes p. To summarize, the above construction determines a map which associates an adele class

$$\mathfrak{a}_g = (-\alpha, (\lambda_p)) \in \mathbb{A}_\mathbb{Q}/\mathbb{Q} = (\mathbb{Q} \otimes_\mathbb{Z} \mathbb{A}_\mathbb{Z})\,/\mathbb{Q} \qquad (1)$$

to a primitive form g with rational Fourier coefficients.

In this paper (see Theorem 1 below), we prove (1) in the case when g is of weight 2. There are advantages to this case. Firstly, there is an abundance of examples since there are infinitely many primitive forms with rational integer Fourier coefficients. Secondly, the infinitude of supersingular primes for such a form proved by Elkies [5] yields a systematic involvement of quantities μ_p.

Theorem 1. *Let g be a weight 2 primitive form. For all but possibly finitely many primes p, the quantities $\lambda_p \in \mathbb{Z}_p$ (and $p\mu_p \in \mathbb{Z}_p$) are p-adic integers.*

Remark 1. Despite the obvious similarity between Proposition 1 and Theorem 1, our proofs of these two statements are based on completely different ideas.

Remark 2. It is not difficult to prove the assertion of Theorem 1 for ordinary primes (Proposition 6 below). However, the argument used in that proof of Proposition 6 does not generalize to other primes.

We derive Theorem 1 from Theorem 2 below. The latter, in our opinion, is an elegant statement of independent interest.

Section 'Proof of Theorem 1' of the paper is devoted to principal ideas involved into the proof of Theorem 1. The proofs of several propositions formulated in Section 'Proof of Theorem 1' are postponed to further sections. Specifically, in Section 'Weak harmonic Maass forms and certain p-adic limits' we recall some facts and definitions related to weak harmonic Maass forms and prove the initial version of Zagier's claim (see Propositions 1, 7). In Section 'Weight zero weak harmonic Maass forms and the pullback of the Weierstrass ζ-function', we relate the weak harmonic Maass form M of weight $2-k=0$ which is good for g to a pullback of the Weierstrass ζ-function. Section 'One-dimensional commutative formal group laws' is devoted to several technical statements on one-dimensional formal group laws. Finally, Section 'The addition law for the Weierstrass ζ-function' is devoted to (elementary) analysis of the addition law for the Weierstrass ζ-function which allows us to make a statement of global nature (i.e. 'for all but possibly finitely many primes').

Methods

Proof of Theorem 1

In order to clarify the ideas of our proof of Theorem 1, we begin with a relation between the theory of weak harmonic Maass forms of weight zero and classical Weierstrass theory of elliptic functions. This relation allows us, in particular, to obtain an interpretation (3) for the quantity α. The assumption $k=2$ is crucial for this discussion.

The modular form g determines a rational elliptic curve along with its modular parametrization (see e.g. [6] for details). Specifically, the map $\Gamma_0(N) \to \mathbb{C}$ defined by

$$\gamma \mapsto \mathcal{E}_g(\gamma(\tau)) - \mathcal{E}_g(\tau)$$

is a group homomorphism, its image is a rank two lattice $\Lambda_g = \Lambda \subset \mathbb{C}$, and Eichler - Shimura theory guarantees that the quantities $g_2 = g_2(\Lambda)$ and $g_3 = g_3(\Lambda)$ defined by

$$g_2(\Lambda) = 60 \sum_{\substack{m\in\Lambda \\ m\neq 0}} m^{-4}, \quad g_3(\Lambda) = 140 \sum_{\substack{m\in\Lambda \\ m\neq 0}} m^{-6}$$

are rational numbers. The quotient $E = \mathbb{C}/\Lambda$ is thus an elliptic curve over $\mathbb{Q}$, and we denote by ω the nowhere vanishing differential on E normalized such that its pullback with respect to the covering map $\psi: \mathbb{C} \to \mathbb{C}/\Lambda$ is dz.

We thus let $X_0(N) = \overline{\Gamma_0(N)\backslash\mathfrak{H}}$ and obtain the map

$$\mathcal{E}_g : X_0(N) \to E = \mathbb{C}/\Lambda$$

such that ω pulls back to $-2\pi i g(\tau)\, d\tau$. Note that we do not consider the minimal model of E here, and thus, in particular, our considerations do not depend on Manin's constant. The map $\mathcal{E}_g$ is defined here as a complex-analytic map, but it is known to be also a birational map between algebraic varieties. In particular, the values of the J-invariant at the preimages of divison points are algebraic numbers. It is well-known that since $\mathcal{E}_g = q + \ldots \in \mathbb{Z}[\![q]\!]$, we have that $\mathcal{E}_g(i\infty) = 0 \in \Lambda$, the origin of E, and the modular parametrization map $\mathcal{E}_g$ is unramified at $i\infty$ (see e.g. [7], Lemma 1).

The Weierstrass ζ-function associated with Λ

$$\zeta(\Lambda, z) = \frac{1}{z} + \sum_{\substack{m\in\Lambda \\ m\neq 0}} \left(\frac{1}{z-m} + \frac{z}{m^2} + \frac{1}{m} \right)$$

is a meromorphic function on $\mathbb{C}$, which is not a lift under ψ of any function on E simply because ζ is not Λ-periodic. However, by a classical observation which goes back to Eisenstein (see [8]), one can make this function Λ-periodic by adding to it a linear combination of z and $\overline{z}$. More precisely, let $a(\Lambda)$ be the volume of the fundamental parallelogram of the lattice Λ. Then the function

$$R(z) := \zeta(\Lambda, z) - \mathbb{S}(\Lambda) z - \frac{\pi}{a(\Lambda)} \overline{z}$$

is Λ-periodic. Here $\mathbb{S}(\Lambda)$ is the value at Λ of the C^∞ Eisenstein series of weight 2

$$\mathbb{S}(\Lambda) = \lim_{s\to 0} \sum_{\substack{m\in\Lambda \\ m\neq 0}} \frac{1}{m^2 |m|^{2s}}.$$

This function is defined in [8] with the help of Eisenstein summation, and has the Fourier expansion (see [9], where the function is denoted by $-\frac{1}{12}S$)

$$\mathbb{S}(2\pi i, 2\pi i\tau) = \frac{1}{4\pi \Im(\tau)} - \frac{1}{12} + 2\sum_{n\geq 1} \sigma_1(n) q^n.$$

Since R is Λ-periodic, it can be pulled back to $X_0(N)$ as

$$N(\tau) := R(\mathcal{E}_g(\tau)).$$

The non-holomorphic $\Gamma_0(N)$- modular function N splits naturally into a sum $N = N^+ + N^-$ of a meromorphic function N^+ and an antiholomorphic function N^-:

$$N^+ = \zeta(\Lambda, \mathcal{E}_g) - \mathbb{S}(\Lambda)\mathcal{E}_g, \qquad N^- = -\frac{\pi}{a(\Lambda)} \overline{\mathcal{E}_g}. \tag{2}$$

In Section 'Weak harmonic Maass forms and certain p-adic limits', we compare the functions M and N, and obtain the following:

Proposition 2. *There exists a positive integer $C \in \mathbb{Z}$ such that*

$$N - CM = N^+ - CM^+ \in \mathbb{Q}((q)).$$

Moreover,

$$N - CM = N^+ - CM^+ \in \mathbb{Z}_p((q))$$

for all but possibly finitely many primes p.

This proposition allows us to indicate a natural choice

$$\alpha = -\frac{\mathbb{S}(\Lambda)}{C} \tag{3}$$

such that $M^+ - \alpha\mathcal{E}_g \in \mathbb{Q}((q))$. Moreover, it allows us to immediately reduce the proof of Theorem 1 to the proof of the following statement.

Theorem 2. *For a primitive cuspform $g = \sum b(n)q^n \in S_2(N)$ with rational integer Fourier coefficients $b(n)$, let*

$$\mathcal{E}_g = \sum_{n\geq 1} \frac{b(n)}{n} q^n$$

be the associated Eichler integral, and let $\Lambda = \Lambda_g$ be the lattice in $\mathbb{C}$ defined above.

For almost every prime p there exist $\lambda_p \in \mathbb{Z}_p$ and $\mu_p \in (1/p)\mathbb{Z}_p$ with $\mu_p = 0$ if $b(p) \not\equiv 0 \mod p$ such that

$$\zeta(\Lambda, \mathcal{E}_g) - \lambda_p \mathcal{E}_g - \mu_p \mathcal{E}_g | V_p \in \mathbb{Z}_p((q)) \tag{4}$$

Proof. The proof of Theorem 2 is based on an algebraic interpretation of the ideas which allowed us to make a Λ-periodic function out of Weierstrass ζ-function. Namely, the elliptic curve E is defined over $\mathbb{Q}$ by the equation

$$E\ :\ y^2 = 4x^3 - g_2 x - g_3, \quad \text{with } x = \wp(\Lambda, z) \text{ and } y = \wp'(\Lambda, z), \tag{5}$$

and carries the differentials of the second kind $\omega = dx/y$ and $\eta = x dx/y$. The fact that the function R above is Λ-periodic can be considered as a consequence of the (complex-analytic) Hodge decomposition $H^1_{dR}(E, \mathbb{C}) = H^{1,0}(E_{\mathbb{C}}) \oplus H^{0,1}(E_{\mathbb{C}})$. Then the value of the weight 2 Eisenstein series $\mathbb{S}(\Lambda)$ becomes the coefficient of ω in the decomposition of the meromorphic differential of the second kind $\eta \in H^1_{dR}(E, \mathbb{C})$ into a linear combination

$$\eta = -\mathbb{S}(\Lambda)\omega - \frac{\pi}{a(\Lambda)}\overline{\omega} \tag{6}$$

of the holomorphic differential $\omega \in H^{1,0}(E_{\mathbb{C}})$ and the antiholomorphic differential $\overline{\omega} \in H^{0,1}(E_{\mathbb{C}})$ (see [9], Section 1.3 for details). Recall that, by definition, meromorphic differentials of the second kind are those which become exact being pulled back to the covering $\mathbb{C} \to E = \mathbb{C}/\Lambda$. In order to clarify the relation to our function R, note that for a meromorphic function $h : \mathbb{C} \to \mathbb{C}$, the differential $h'(z)\, dz$ is of the second kind if and only if the function in two complex variables

$$\delta(h)(z_1, z_2) := h(z_1 + z_2) - h(z_1) - h(z_2)$$

is Λ^2-periodic. This differential is exact if and only if h itself is Λ-periodic. A standard addition formula for the Weierstrass ζ-function (see (22)) implies that $\delta(\zeta)$ is Λ^2-periodic. For both functions z and $\overline{z}$ we have that $\delta(z) = \delta(\overline{z}) = 0$; therefore, these functions are Λ^2-periodic trivially. We can now interpret both the Hodge decomposition (6) and the Λ-periodicity of R as a decomposition of the meromorphic function $\Psi(z) (= \zeta(\Lambda, z))$ such that $\delta(\Psi)$ is Λ^2-periodic into a linear combination of z and $\overline{z}$ modulo Λ-periodic functions. Note that Λ-periodic functions are exactly those which pull back from E. We denote the linear space of meromorphic Λ-periodic functions with poles outside Λ by $\mathcal{B}$. While $\zeta(\Lambda, z)$ has its poles in Λ, its shift by a Λ-periodic function $\zeta(\Lambda, z) + \wp'(\Lambda, z)/2\wp(\Lambda, z)$ has its poles outside Λ.

This discussion allows us to rewrite the Λ-periodicity of R (or, equivalently, (6)) as

$$\zeta(\Lambda, z) + \frac{\wp'(\Lambda, z)}{2\wp(\Lambda, z)} \equiv \mathbb{S}(\Lambda) z + \frac{\pi}{a(\Lambda)}\overline{z} \mod \mathcal{B}, \tag{7}$$

and motivates the following definition for the first de Rham cohomology of a formal group law. For a one-dimensional formal group law $F(X, Y)$, defined over $\mathbb{Z}_p$ (see Section 'One-dimensional commutative formal group laws' for basic definitions and properties), a formal power series $\xi \in X\mathbb{Q}_p[\![X]\!]$ is called a cocycle if

$$\delta_F(\xi)(X, Y) := \xi(F(X, Y)) - \xi(X) - \xi(Y) \in \mathbb{Z}_p[\![X, Y]\!] \otimes \mathbb{Q}_p,$$

and ξ is called a coboundary if $\xi(X) \in \mathbb{Z}_p[\![X]\!] \otimes \mathbb{Q}_p$. The first de Rham cohomology $H^1_{dR}(F)$ is defined as the quotient of cocycles modulo coboundaries.

This definition introduces $H^1_{dR}(F)$ as a vector space over $\mathbb{Q}_p$ while Theorem 2 requires us to consider a related $\mathbb{Z}_p$-module. In Section 'Weight zero weak harmonic Maass forms and the pullback of the Weierstrass ζ-function', we prove the following proposition which helps us to choose a natural normalization for that.

Proposition 3. *Let F be a one-dimensional formal group law over $\mathbb{Z}_p$, and let $\xi \in H^1_{dR}(F)$ be a cocycle. Then $\xi'(X) \in \mathbb{Z}_p[\![X]\!] \otimes \mathbb{Q}_p$.* □

Proposition 3 suggests us to consider the $\mathbb{Z}_p$-module $\mathbb{D}(F) \subset H^1_{dR}(F)$ defined by

$$\mathbb{D}(F) = \frac{\left\{\xi \in X\mathbb{Q}_p[\![X]\!] \mid \xi' \in \mathbb{Z}_p[\![X]\!] \text{ and } \xi(F(X,Y)) - \xi(X) - \xi(Y) \in \mathbb{Z}_p[\![X,Y]\!]\right\}}{\left\{\xi \in X\mathbb{Z}_p[\![X]\!]\right\}}.$$

This module was considered by N. Katz in [10]. In particular, by Theorem 5.3.3 of [10], $\mathbb{D}(F)$ is a free $\mathbb{Z}_p$-module of rank $h = \text{height}(F^{(p)})$, where $F^{(p)}$ is the reduction of F modulo p, assuming that $h < \infty$. As an example, for such a formal group law F, its logarithm $\ell_F \in \mathbb{D}(F)$ by Chapter IV, Proposition 5.5 of [11], and thus obviously spans $\mathbb{D}(F)$ if $h = 1$. We will also need an explicit basis of $\mathbb{D}(F)$ if $h = 2$.

Proposition 4. *Let F be a one-dimensional formal group law over $\mathbb{Z}_p$ such that its modulo p reduction $F^{(p)}$ is of height $h = 1$ or 2. Let $\ell_F(X) \in \mathbb{Q}_p[\![X]\!]$ be the logarithm of F.*

When $h = 1$, the one-dimensional $\mathbb{Z}_p$-module $\mathbb{D}(F)$ is spanned by $\ell_F(X)$.

When $h = 2$, the two-dimensional $\mathbb{Z}_p$-module $\mathbb{D}(F)$ is spanned by $\ell_F(X)$ and $p^{-1}\ell_F(X^p)$.

We provide an elementary proof of Proposition 4 for $h = 2$ in Section 'Weight zero weak harmonic Maass forms and the pullback of the Weierstrass ζ-function' of the paper. This proof generalizes to any $h < \infty$, though we do not need and do not prove any generalization here. This proposition does not seem to be new. For instance, it was pointed out by the referee that Proposition 4 should follow from the fact proved by N. Katz in [10] that in the supersingular case the Dieudonné module gives the whole H^1_{crys} along with the well-known fact that the Frobenius map is bijective on H^1_{crys}.

We are particularly interested here in the the formal group law D_g over $\mathbb{Z} \hookrightarrow \mathbb{Z}_p$ whose logarithm is $\mathcal{E}_g(q) \in \mathbb{Q}[\![q]\!] \hookrightarrow \mathbb{Q}_p[\![q]\!]$ for a primitive form $g \in S_2(N)$ of weight 2 with rational integral Fourier coefficients. For a prime $p \nmid N$, the height of the modulo p reduction of D_g is 1 if $b(p) \neq 0$, and 2 otherwise. We thus obtain that

$$\text{the } \mathbb{Z}_p\text{-basis of } \mathbb{D}(D_g) = \begin{cases} \mathcal{E}_g & \text{if } b(p) \neq 0 \\ \mathcal{E}_g, \quad p^{-1}\mathcal{E}_g|V & \text{if } b(p) = 0. \end{cases} \tag{8}$$

Theorem 2 would follow from (8) immediately if $\zeta(\Lambda, \mathcal{E}_g)$ was an element of $\mathbb{D}(D_g)$. That however is not the case at least because $\zeta(\Lambda, \mathcal{E}_g) = 1/q + \ldots \in \mathbb{Q}(\!(q)\!) \not\subseteq \mathbb{Q}_p[\![q]\!]$.

We now consider the one-dimensional formal group law $\hat{E}$ determined by Equation 5 (see Section 'Weight zero weak harmonic Maass forms and the pullback of the Weierstrass ζ-function' for a definition) with the formal group parameter $X = -2x/y = -2\wp(\Lambda, z)/\wp'(\Lambda, z)$. Since $X \in z\mathbb{Q}[\![z]\!]$, one can write $\zeta(\Lambda, z) = 1/X + C_0 + \Phi(X)$ with $C_0 \in \mathbb{Q}$ and a formal power series $\Phi(X) \in X\mathbb{Q}[\![X]\!]$.

It will be convenient for us to introduce the following notation. Throughout, we write $\mathcal{Z}$ for a subring of $\mathbb{Q}$ which coincides with $\mathbb{Z}[1/l]$ for some integer l. (For example, a power series $\phi = \sum u(n)q^n \in \mathcal{Z}[\![q]\!]$ means that $u(n) \in \mathbb{Z}_p$ for all integers n for all but possibly finitely many primes p.)

In Section 'One-dimensional commutative formal group laws', we employ the addition law for the Weierstrass ζ-function to prove that the formal power series $\Phi(X)$ satisfies

$$\Phi(\hat{E}(X,Y)) - \Phi(X) - \Phi(Y) \in \mathcal{Z}[\![X,Y]\!]. \tag{9}$$

In Proposition 10, we show that the power series $f := -2\wp(\Lambda, \mathcal{E}_g)/\wp'(\Lambda, \mathcal{E}_g) \in \mathcal{Z}[\![q]\!]$ establishes an isomorphism $f : D_g \to \hat{E}$ over $\mathcal{Z}$. This proposition is a simplified version of a more precise theorem of Honda [12] (cf. also [13]). This isomorphism allows us to rewrite (9) as

$$\delta_{D_g}\left(\zeta(\Lambda, \mathcal{E}_g) + \frac{\wp'(\Lambda, \mathcal{E}_g)}{2\wp(\Lambda, \mathcal{E}_g)}\right)(q_1, q_2) \in \mathcal{Z}[\![q_1, q_2]\!],$$

and to use Proposition 9 below in order to conclude that

$$\frac{d}{dq}\left(\zeta(\Lambda, \mathcal{E}_g) + \frac{\wp'(\Lambda, \mathcal{E}_g)}{2\wp(\Lambda, \mathcal{E}_g)}\right) \in \mathcal{Z}[\![q]\!].$$

We thus conclude that

$$\zeta(\Lambda, \mathcal{E}_g) + \frac{\wp'(\Lambda, \mathcal{E}_g)}{2\wp(\Lambda, \mathcal{E}_g)} \in \mathbb{D}(D_g) \tag{10}$$

for all but possibly finitely many primes p. We take into the account that (see Proposition 10) $-2\wp(\Lambda, \mathcal{E}_g)/\wp'(\Lambda, \mathcal{E}_g) = q + \ldots \in \mathcal{Z}[\![q]\!]$, therefore the reciprocal

$$\frac{\wp'(\Lambda, \mathcal{E}_g)}{2\wp(\Lambda, \mathcal{E}_g)} \in \mathcal{Z}(\!(q)\!).$$

This observation combined with (10) and (8) accomplishes our proof of Theorem 2.

As it was mentioned above, Theorem 1 follows from Theorem 2.

Results

Weak harmonic Maass forms and certain *p*-adic limits

In this section, $g \in S_k(N)$ is a primitive form with rational integral Fourier coefficients of arbitrary even integer weight $k \geq 2$. In a moment, we will pay special attention to the cases when $N = 1$, and $\dim S_k(1) = 1$, namely $k = 12, 16, 18, 20, 22, 26$. In particular, Zagier's initial claim, namely Proposition 1 which motivated this project and served as its starting point, is a special case of Proposition 7 which is proved in this section.

To begin with, we recall briefly the definition of and some basic facts about weak harmonic Maass forms. For further details, we refer the reader to [2,14]. For $\tau \in \mathfrak{H}$, the complex upper half-plane, let $\tau = x + iy$ with $x, y \in \mathbb{R}$. Let Δ_{2-k} be the weight $2-k$ hyperbolic Laplacian

$$\Delta_{2-k} := -y^2\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) + i(2-k)y\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right).$$

A harmonic weak Maass form M of weight $2-k$ and level N is a smooth function on the upper half-plane which satisfies the following properties:

1. For all $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(N)$ we have
$$M\left(\frac{a\tau + b}{c\tau + d}\right) = (c\tau + d)^{2-k} M(\tau).$$

2. We have that $\Delta_{2-k}(M) = 0$.
3. The function $M(\tau)$ has at most linear exponential growth at all cusps of $\Gamma_0(N)$.

We denote the $\mathbb{C}$-vector space of weak harmonic Maass forms by $H_{2-k}(N)$.

The operator

$$\xi = 2iy^{2-k}\overline{\frac{\partial}{\partial\bar{\tau}}} \; : \; H_{2-k}(N) \to M_k^!(N)$$

takes weak harmonic Maass forms to weakly holomorphic (i.e. holomorphic on the upper half-plane with possible poles at cusps) modular forms of weight k and level N.

We now restrict our attention to the subspace of weak harmonic Maass forms which map to cusp forms in $S_k(N)$ under the ξ operator. Following [2], we say that a harmonic weak Maass form $M \in H_{2-k}(N)$ is good for g if the following conditions are satisfied:

(i) The principal part of M at the cusp ∞ belongs to $\mathbb{Q}[q^{-1}]$.
(ii) The principal parts of M at other cusps of $\Gamma_0(N)$ are constant.
(iii) We have that $\xi(M) = \|g\|^{-2}g$, where $\|\cdot\|$ is the usual Petersson norm.

The existence of $M \in H_{2-k}(N)$ which is good for g is proved in [2]. Every weak harmonic Maass form M decomposes naturally into the sum

$$M = M^+ + M^-,$$

where the function M^+ is holomorphic on $\mathfrak{H}$ and meromorphic at cusps. The function M^+ is referred to as a mock modular form. If M is good for g, then g is called the shadow of M^+. The mock modular form M^+ has a Fourier expansion in $q = e^{2\pi i\tau}$, and the coefficients of this expansion are the subject of interest in this paper. Theorem 1.1 of [3] guaranties the existence of $\alpha \in \mathbb{R}$ such that

$$\mathcal{M}_\alpha := M^+ - \alpha\mathcal{E}_g \in \mathbb{Q}((q)).$$

Let p be a prime. For a formal power series series $\phi = \sum c(n)q^n$ with rational coefficients $c(n) \in \mathbb{Q} \subset \mathbb{Q}_p$ we put

$$\mathrm{ord}_p(\phi) = \min_n \mathrm{ord}_p(c(n)),$$

and introduce the metrics on the set of formal power series ϕ such that $\mathrm{ord}_p(\phi) > -\infty$ by putting $\|\phi\|_p := p^{-\mathrm{ord}_p(\phi)}$. We tacitly identify rational numbers with elements of $\mathbb{Q}_p$ under the natural embedding $\mathbb{Q} \hookrightarrow \mathbb{Q}_p$.

Proposition 5. *Assume the notations and conventions above. If $k > 2$, we assume additionally that $p^2|N$ for all primes $p|N$.*

For every prime p there exist $\lambda_p, \mu_p \in \mathbb{Q}_p$ such that

$$\mathit{ord}_p\left(\mathcal{M}_\alpha - \lambda_p\mathcal{E}_g - \mu_p\mathcal{E}_g|V\right) > -\infty.$$

The quantities $\lambda_p, \mu_p \in \mathbb{Q}_p$ are defined uniquely if $p^2 \nmid N$.

Proof. The argument below is a refinement of similar arguments from the proof of Theorem 1.1 in [4] and the proof of Theorem 1.2 in [3] adapted for our current purposes.

Recall that

$$g = \sum_{n\geq 1} b(n)q^n \in S_k(N).$$

The action of the Hecke operators on the weak harmonic Maass forms of weight $2-k$ is described in [2]. For a prime p, the Hecke operator T_p acting on forms of weight $2-k$ is defined by

$$T_p = \begin{cases} U + p^{1-k}V & \text{if } p \nmid N \\ U & \text{if } p \mid N \end{cases}, \tag{11}$$

and it is not difficult to show (see [2], proof of Theorem 1.3) that, since M is good for g,

$$\mathcal{M}_\alpha|_{2-k}T_p = p^{1-k}b(p)\mathcal{M}_\alpha + R_p \tag{12}$$

with a weakly holomorphic modular form $R_p \in M^!_{2-k}(N) \cap \overline{\mathbb{Q}}((q))$. Equation 12 is an analog of the obvious

$$\mathcal{E}_g|_{2-k}T_p = p^{1-k}b(p)\mathcal{E}_g.$$

We firstly consider the case when $p^2 \mid N$. It follows from Theorem 4.6.17 (3) of [1], that $b(p) = 0$, thus $\mathcal{M}_\alpha|U \in M^!_{2-k}(N)$; therefore, $\mathrm{ord}_p(\mathcal{M}_\alpha|U) > -\infty$. At the same time, by Bol's identity, $D^{k-1}(\mathcal{M}_\alpha) \in M^!_k(N)$, where $D := (2\pi i)^{-1}d/d\tau$, and we conclude that also $\mathrm{ord}_p(\mathcal{M}_\alpha) > -\infty$. It follows that $\lambda_p = \mu_p = 0$ does the job in this case.

Since now on, we assume that $p^2 \nmid N$. The uniqueness clause of Proposition 5 follows immediately from $\mathrm{ord}_p(\mathcal{E}_g) = -\infty$ (and therefore, $\mathrm{ord}_p(\mathcal{E}_g|V) = -\infty$) in this case by ([1], Theorem 4.6.17). We thus only need to prove the existence clause. Moreover, we can and will assume that $k = 2$ if $p \mid N$.

We now introduce certain quantities β and β'. Our definition depends on whether $p \mid N$ or $p \nmid N$.

If $p \mid N$ then by ([1], Theorem 4.6.17 (2)), we have $b(p)^2 = p^{k-2}$, and we assumed that $k = 2$; therefore, $b(p) = \pm 1$. In this case, we put $\beta = b(p)$ and $\beta' = p\beta$.

If $p \nmid N$ denote by β and β' the roots of the p-th Hecke polynomial

$$Z^2 - b(p)Z + p^{k-1} = (Z-\beta)(Z-\beta')$$

ordered such that $\mathrm{ord}_p(\beta) \leq \mathrm{ord}_p(\beta')$.

Note that in both cases $\beta\beta' = p^{k-1}$, thus $\mathrm{ord}_p(\beta) + \mathrm{ord}_p(\beta') = k-1 > 0$, and therefore $\mathrm{ord}_p(\beta') > 0$.

Put

$$\mathcal{G} = \begin{cases} \mathcal{M}_\alpha - \frac{1}{\beta}\mathcal{M}_\alpha|V & \text{if } p \nmid N \\ \mathcal{M}_\alpha & \text{if } k = 2, p \mid N, \text{ and } p^2 \nmid N \end{cases} \tag{13}$$

It follows from (11) and (12) that

$$\mathcal{G}|U = \frac{1}{\beta'}\mathcal{G} + R_p.$$

Iterating the action of the U-operator, we produce for $l \geq 1$

$$\beta'^l\mathcal{G}|U^l = \mathcal{G} + \beta'\sum_{m=0}^{l-1}\beta'^m R_p|U^m \tag{14}$$

We now consider separately p-ordinary (i.e. $\mathrm{ord}_p(\beta) = 0$) and non-$p$-ordinary (i.e. $\mathrm{ord}_p(\beta) > 0$) cases.

We begin with the p-ordinary case. In this case $\beta, \beta' \in \mathbb{Q}_p$ by Hensel's lemma if $p \nmid N$, and by the definition above if $p \mid N$.

Since $R_p \in M^!_{2-k}(N)$, we have that $\mathrm{ord}_p(R_p) > -\infty$. Thus, the limit $\lim_{l\to\infty} \beta'^l \mathcal{G}|U^l$ exists in $\mathbb{Q}_p[\![q]\!]$, since $\mathrm{ord}_p(\beta') = k-1 > 0$.

Since $U = U_p$ commutes with Hecke operators T_m, for all primes $m \neq p$, a standard induction argument proves that there exist $u \in \mathbb{Q}_p$ such that

$$\lim_{l\to\infty} \beta'^l \mathcal{G}|U^l = \begin{cases} u\left(\mathcal{E}_g - \frac{1}{\beta}\mathcal{E}_g|V\right) & \text{if } p \nmid N \\ u\mathcal{E}_g & \text{if } k=2, p \mid N, \text{and } p^2 \nmid N. \end{cases} \tag{15}$$

We now consider the cases when $p \nmid N$ and $p \mid N$ separately. We claim that in either case, $\lambda_p = u$ and $\mu_p = 0$ do the job.

We begin with the case when $p \mid N$ (thus, $k = 2$ and $p^2 \nmid N$). Equations 14 and 15 imply that

$$\mathcal{M}_\alpha - u\mathcal{E}_g = \mathcal{G} - u\mathcal{E}_g = -\beta' \sum_{l\geq 0} \beta'^l R_p|U^l.$$

Since $\mathrm{ord}_p(R_p) > -\infty$, we conclude that $\mathrm{ord}_p\left(\sum_{l\geq 0} \beta'^l R_p|U^l\right) > -\infty$ as required.

We now consider the case when $p \nmid N$. In this case, equations (14) and (15) imply that

$$\mathcal{G} - u\left(\mathcal{E}_g - \frac{1}{\beta}\mathcal{E}_g|V\right) = -\beta' \sum_{l\geq 0} \beta'^l R_p|U^l,$$

thus

$$\left(\mathcal{M}_\alpha - u\mathcal{E}_g\right)|\left(1 - \frac{1}{\beta}V\right) = -\beta' \sum_{l\geq 0} \beta'^l R_p|U^l.$$

Again, since $\mathrm{ord}_p(R_p) > -\infty$, we conclude that $\mathrm{ord}_p\left(\sum_{l\geq 0} \beta'^l R_p|U^l\right) > -\infty$. We now invert the operator $\left(1 - \frac{1}{\beta}V\right)^{-1} = \sum_{m\geq 0} \beta^{-m}V^m$, which makes sense in the q-adic topology on $\mathbb{Q}_p(\!(q)\!)$, put

$$\lambda_p = u \quad \text{and} \quad \mu_p = 0,$$

and derive that $\mathrm{ord}_p\left(\mathcal{M}_\alpha - \lambda_p\mathcal{E}_g\right) > -\infty$ as claimed.

We now consider the non-p-ordinary case, and assume that $\mathrm{ord}_p(\beta) > 0$. Our assumptions along with the discussion above imply that $\mathrm{ord}_p(\beta) > 0$ may happen only if $p \nmid N$. In this case, we can only guarantee that $\beta, \beta' \in \mathbb{Q}_p(\sqrt{-p})$. We now consider two similar formal power series $\mathcal{G}'$ and $\mathcal{G}$:

$$\mathcal{G} = \mathcal{M}_\alpha - \frac{1}{\beta}\mathcal{M}_\alpha|V \quad \text{and} \quad \mathcal{G}' = \mathcal{M}_\alpha - \frac{1}{\beta'}\mathcal{M}_\alpha|V$$

Note that this definition of $\mathcal{G}$ is consistent with (13) because $p \nmid N$ in this case. As above, it follows from (11) and (12) that

$$\mathcal{G}|U = \frac{1}{\beta'}\mathcal{G} + R_p \quad \text{and} \quad \mathcal{G}'|U = \frac{1}{\beta}\mathcal{G}' + R_p,$$

and iterating the action of the U-operator we obtain an analog of (14)

$$\beta'^l\mathcal{G}|U^l = \mathcal{G} + \beta' \sum_{m=0}^{l-1} \beta'^m R_p|U^m \quad \text{and} \quad \beta^l\mathcal{G}'|U^l = \mathcal{G}' + \beta \sum_{m=0}^{l-1} \beta^m R_p|U^m, \tag{16}$$

and both infinite sums converge since both $\mathrm{ord}_p(\beta) > 0$ and $\mathrm{ord}_p(\beta') > 0$ while still $\mathrm{ord}_p(R_p) > -\infty$. An argument identical to that in the p-ordinary case now implies the existence of two quantities $u, v \in \mathbb{Q}_p(\sqrt{-p})$ such that

$$\lim_{l\to\infty} \beta'^l \mathcal{G}|U^l = u\left(\mathcal{E}_g - \frac{1}{\beta}\mathcal{E}_g|V\right) \quad \text{and} \quad \lim_{l\to\infty} \beta^l \mathcal{G}'|U^l = v\left(\mathcal{E}_g - \frac{1}{\beta'}\mathcal{E}_g|V\right). \qquad (17)$$

It follows from (16) and (17) that

$$\mathcal{G} - u\left(\mathcal{E}_g - \frac{1}{\beta}\mathcal{E}_g|V\right) = -\beta'\sum_{l\geq 0}\beta'^l R_p|U^l \text{ and } \mathcal{G}' - v\left(\mathcal{E}_g - \frac{1}{\beta'}\mathcal{E}_g|V\right) = -\beta\sum_{l\geq 0}\beta^l R_p|U^l,$$

and a linear combination of these two equations transforms to

$$\mathcal{M}_\alpha - \frac{u\beta - v\beta'}{\beta-\beta'}\mathcal{E}_g - \frac{v-u}{\beta-\beta'}\mathcal{E}_g|V = p^{k-1}\sum_{l\geq 0}\frac{\beta^l - \beta'^l}{\beta-\beta'}R_p|U^l.$$

We now notice that the p-adic valuation of the series on the right is finite and put

$$\lambda_p = \frac{u\beta - v\beta'}{\beta-\beta'}, \qquad \mu_p = \frac{v-u}{\beta-\beta'}. \qquad (18)$$

We still have to show that both these quantities actually belong to $\mathbb{Q}_p$, not merely $\mathbb{Q}_p(\sqrt{-p})$. That follows from their definition along with the consideration of the action of the Galois group $Gal(\mathbb{Q}_p(\sqrt{-p})/\mathbb{Q}_p)$ on (17). □

The following proposition was proved by Zagier; this proposition motivates the claim about the adele.

Proposition 6. *For all but possibly finitely many primes p such that g is p-ordinary, we have $\mathrm{ord}_p(\lambda_p) \geq 0$.*

Proof. Let

$$\mathcal{M}_\alpha = \sum_{n\gg-\infty} a(n)q^n.$$

We have that

$$D^{k-1}(\mathcal{M}_\alpha) = \sum_{n\gg-\infty} n^{k-1}a(n)q^n \in M_k^!(N),$$

therefore $\mathrm{ord}_p\left(D^{k-1}(\mathcal{M}_\alpha)\right) \geq 0$, and, in particular, for all $l \geq 0$

$$l(k-1) + \mathrm{ord}_p(a(p^l)) \geq 0 \qquad (19)$$

for all but possibly finitely many primes p. Since g is p-ordinary, $\lambda_p = u$ and $\mathrm{ord}_p(\beta') = k-1$. We calculate the coefficient of q in both sides of (17) and obtain that

$$\lambda_p = u = \lim_{l\to\infty}\beta'^l\left(a(p^l) - \frac{1}{\beta}a\left(p^{l-1}\right)\right). \qquad (20)$$

The proposition follows from (19) combined with (20) and the ultrametric inequality. □

The proof of Proposition 6 does not generalize to non-ordinary primes, and in no case the set of non-ordinary primes is known to be finite. However, in the case $g = \Delta \in S_{12}(1)$ (and in some similar cases), there is an easy argument independent on whether a prime is ordinary or not. Specifically, we now prove the following proposition which mildly generalizes Proposition 1.

Proposition 7. *If $N = 1$ and $k = 12, 16, 18, 20, 22, 26$, then for all but possibly finitely many primes p we have that $\lambda_p \in \mathbb{Z}_p$, and $p^{k-1}\mu_p \in \mathbb{Z}_p$.*

Remark 3. The case $N = 1$ and $g = \Delta$ of weight $k = 12$ was considered by Zagier. It was his observation that $(-\alpha, (\lambda_p))$ is an adele of $\mathbb{Q}$. Our statement about μ_p here does not seriously enhance Proposition 7. Firstly, we do not know whether there are infinitely many primes p such that $p \mid b(p)$ and therefore μ_p may be non-zero. Secondly, the quantities μ_p are obviously independent on the choice of α, therefore we do not see any natural way to speak about an adele class determined by these quantities. Although our proof of Proposition 7 may be generalized to some other cases, we do not know any interesting and natural infinite series of examples which may be treated using this kind of argument.

Proof. In these cases, $\dim S_k(1) = 1$, and thus, we may and will assume that $\mathcal{M}_\alpha$ has a principal part of q^{-1}. Let

$$R_p = \sum_{n \gg -\infty} d_p(n) q^n.$$

In the p-ordinary case, we equate the coefficients of q in (14) and take into the account (15) to obtain

$$\lambda_p = a(1) + \beta' \sum_{l \geq 0} \beta'^l d_p\left(p^l\right).$$

In the non-p-ordinary case, we equate the coefficients of q in (16) and take into the account (17) and (18) to obtain

$$\lambda_p = a(1) - p^{k-1} \sum_{l \geq 0} \frac{\beta^l - \beta'^l}{\beta - \beta'} d_p\left(p^l\right), \qquad \mu_p = \sum_{l \geq 0} \frac{\beta^{l+1} - \beta'^{l+1}}{\beta - \beta'} d_p\left(p^l\right).$$

It suffices to show that $\mathrm{ord}_p(d_p(p^l)) + k - 1 \geq 0$ for all $l \geq 0$ for almost all primes p. That will obviously follow from the inequality

$$\mathrm{ord}_p\left(p^{k-1} R_p\right) \geq 0$$

for all primes $p > 3$. In order to verify this inequality, note that since the principal part of $\mathcal{M}_\alpha$ is q^{-1}, the equations (11) and (12) imply that the principal part of $p^{k-1}R_p$ is $q^{-p} - b(p)q^{-1}$. Thus, the modular form $\Delta^p p^{k-1} R_p \in M_{12p+2-k}$ has its coefficients of $q^0, q^1, \ldots, q^{p-1}$ in $\mathbb{Z}$. Since

$$\dim M_{12p+2-k} = \left\lfloor \frac{12p + 2 - k}{12} \right\rfloor + 1 \leq p,$$

it follows from Theorem X.4.2 of [15], that all coefficients of $\Delta^p p^{k-1} R_p \in \mathbb{Z}[1/6][\![q]\!]$, and, therefore, of $p^{k-1} R_p \in \mathbb{Z}[1/6][\![q]\!]$ as required. $\square$

Weight zero weak harmonic Maass forms and the pullback of the Weierstrass ζ-function

In this section, we prove Proposition 2. We thus assume that $k = 2$ and preserve all previous notations and conventions throughout this section. Recall that $M = M^+ + M^- \in H_0(N)$ is a weak harmonic Maass form which is good for $g \in S_2(N)$. Since constants belong to $H_0(N)$, we may and will always assume that the constant term of the Fourier expansion at infinity of the mock modular form M^+ is a rational number. Note that the weight zero Laplacian Δ_0 is, up to a factor of y^2, classical and, since $k = 2$, the condition $\Delta_0(M) = 0$ simply means that M is a harmonic function. The decomposition $M = M^+ + M^-$ simplifies to the decomposition of a harmonic function into the sum of meromorphic and anti-holomorphic functions. We begin with a comparison of the anti-holomorphic parts. Since M is good for g, we have that

$$\xi(M) = 2i\overline{\frac{\partial M}{\partial \overline{\tau}}} = 2i\overline{\frac{\partial M^-}{\partial \overline{\tau}}} = \frac{g}{\|g\|^2}.$$

On the other hand, (2) implies that

$$2i\overline{\frac{\partial N}{\partial \overline{\tau}}} = 2i\overline{\frac{\partial N^-}{\partial \overline{\tau}}} = \frac{4\pi^2}{a(\Lambda)}g.$$

A substitution calculation (see e.g. [6], p.374) implies that

$$\|g\|^2 = \frac{1}{4\pi^2}Ca(\Lambda),$$

where $C \in \mathbb{Z}$ is the degree of the map $\mathcal{E}_g \ : X_0(N) \to \mathbb{C}/\Lambda$. We thus conclude that $N^- = CM^-$, and therefore the function

$$N^+ - CM^+ = N - CM$$

is a meromorphic $\Gamma_0(N)$-invariant function on $\overline{\mathfrak{H}} = \mathfrak{H} \cup$ cusps. Note that while M^+ has its only pole at infinity, N^+ (and thus $N^+ - CM^+$) may have poles both at other cusps and in the interior of the upper half-plane. However, since the map $X_0(N) \to E$ is algebraic, N may have only finitely many poles τ_i of multiplicities κ_i in the fundamental domain $\Gamma_0(N)\backslash\mathfrak{H}$, and the values $J(\tau_i) \in \overline{\mathbb{Q}}$ of the J-invariant at these poles are algebraic numbers. It follows that $(N - CM)\prod_i(J(\tau) - J(\tau_i))^{\kappa_i} \in M_0^!(N)$ is a weakly holomorphic modular form with algebraic coefficients of the principal parts of its its Fourier expansion at all cusps. Therefore, by Chapter 6.2 in [16], we have that the Fourier expansion at infinity $(N - CM)\prod_i(J(\tau) - J(\tau_i))^{\kappa_i} \in \overline{\mathbb{Q}}((q))$, and dividing by back by $\prod_i(J(\tau) - J(\tau_i))^{\kappa_i}$ we conclude that $N^+ - CM^+ = N - CM \in \overline{\mathbb{Q}}((q))$ with $\mathrm{ord}_p(N^+ - CM^+) > -\infty$ for all but possibly finitely many primes p. That will imply the second claim of Proposition 2 if we show that $N^+ - CM^+ = N - CM \in \mathbb{Q}((q))$, or, equivalently, that $\alpha + \mathbb{S}(\Lambda)/C \in \mathbb{Q}$. We note that $\mathcal{M}_\alpha = M^+ - \alpha\mathcal{E}_g \in \mathbb{Q}((q))$ and $N^+ - \frac{\mathbb{S}(\Lambda)}{C}\mathcal{E}_g \in \mathbb{Q}((q))$ by (2), take into the account that $\mathrm{ord}_p(\mathcal{E}_g) = -\infty$ for almost all primes, and conclude that $\alpha + \mathbb{S}(\Lambda)/C \in \mathbb{Q}$.

One-dimensional commutative formal group laws

In this section, we recall some basics on commutative formal group laws, prove Propositions 3 and 4, and establish a simplified explicit version of Honda's theorem in Proposition 10. Recall (see, e.g. [11], Chapter IV.2) that, for a commutative ring $\mathcal{R}$ with an identity element, a one-dimensional commutative formal group law F over $\mathcal{R}$ is a power series $F(X, Y) \in \mathcal{R}[\![X, Y]\!]$ which satisfies the following (not actually independent) conditions.

(a) $F(X, Y) = X + Y +$ (terms of degree ≥ 2)
(b) $F(X, F(Y, Z)) = F(F(X, Y), Z)$
(c) $F(X, Y) = F(Y, X)$.
(d) There is a unique power series $\iota(X) \in \mathcal{R}[\![X]\!]$ such that $F(X, \iota(X)) = 0$.
(e) $F(X, 0) = X$ and $F(0, Y) = Y$.

All formal group laws considered in this paper are one-dimensional and commutative, and we will skip these adjectives.

A ring embedding $\mathcal{R}_1 \hookrightarrow \mathcal{R}_2$ allows one to consider a formal group law over $\mathcal{R}_1$ as a formal group law over $\mathcal{R}_2$, and we do that tacitly using the ring embeddings $\mathbb{Z} \hookrightarrow \mathcal{Z} \hookrightarrow \mathbb{Z}_p$ for almost all primes p throughout.

We begin with Proposition 3.

Proof. Let $\xi \in X\mathbb{Q}_p[\![X]\!]$ be a cocycle of F so that

$$\delta(\xi)(X,Y) = \xi(F(X,Y)) - \xi(X) - \xi(Y) \in p^{-a}\mathbb{Z}_p[\![X,Y]\!]$$

for an integer a. We differentiate with respect to Y and set $Y = 0$ to obtain that

$$\xi'(F(X,0))F_2(X,0) - \xi'(0) \in p^{-a}\mathbb{Z}_p[\![X]\!],$$

where $F_2(X,Y) \in \mathbb{Z}_p[\![X,Y]\!]$ is the partial derivative of $F(X,Y)$ with respect to Y. Since $F(X,0) = X$ and $F_2(X,0) \in 1 + \mathbb{Z}_p[\![X]\!]$,

$$\xi'(X) \in \left(\xi'(0) + p^{-a}\mathbb{Z}_p[\![X]\!]\right)/F_2(X,0) \subseteq p^{-t}\mathbb{Z}_p[\![X]\!]$$

with $t = \max(a,b)$ and $b \in \mathbb{Z}$ such that $p^b\xi'(0) \in \mathbb{Z}_p$ as required. □

We record the outcome of the above argument in the special case $a = b = t = 0$ as a separate proposition.

Proposition 8. *Let F be a formal group law over $\mathbb{Z}_p$. If $\xi \in X\mathbb{Q}_p[\![X]\!]$ satisfies $\delta(\xi)(X,Y) \in \mathbb{Z}_p[\![X,Y]\!]$ and $ord_p(\xi'(0)) = 0$, then $\xi'(X) \in \mathbb{Z}_p[\![X]\!]$.*

Although the following proposition has a global nature, it follows immediately from Proposition 8.

Proposition 9. *Let F be a one-dimensional formal group over $\mathcal{Z}$, and let $\xi \in X\mathbb{Q}[\![X]\!]$ satisfy $\delta(\xi)(X,Y) \in \mathcal{Z}[\![X,Y]\!]$. Then $\xi'(X) \in \mathcal{Z}[\![X]\!]$.*

Proof. Indeed, since $\xi'(0) \in \mathbb{Q}$, we have that $\mathrm{ord}_p(\xi'(0)) = 0$ for almost all primes p, and therefore, by Proposition 8, $\xi'(X) \in \mathbb{Z}_p[\![X]\!]$ for almost all primes p, and our claim follows from that. □

Let F, G be two formal group laws. A power series $f \in X\mathcal{R}[\![X]\!]$ is called a homomorphism from F to G if

$$f(F(X,Y)) = G(f(X), f(Y)).$$

For example, for a formal group F over the ring $\mathbb{Z}/p\mathbb{Z}$ the power series $Fr_p(X) := X^p$ is an endomorphism of F, and it is called Frobenius endomorphism.

A homomorphism f is called a weak isomorphism if it has a two-sided inverse. The inverse, if it exists, is given by the formal power series f^{-1} such that

$$f^{-1}(f(X)) = f(f^{-1}(X)) = X.$$

Additionally, f is called a strong isomorphism if $f'(0) = 1$. We will deal only with strong isomorphisms here, and therefore drop the adjective. It is easy to check that a homomorphism f which satisfies $f'(0) = 1$ is an isomorphism. Equivalently, for a power series $f \in X\mathcal{R}[\![X]\!]$ such that $f'(0) = 1$ there exists $f^{-1} \in X\mathcal{R}[\![X]\!]$. If $\mathcal{R}$ is a $\mathbb{Q}$-algebra, then any formal group law F over $\mathcal{R}$ is isomorphic to the additive group $\mathbb{G}_a(X,Y) := X + Y$. This isomorphism $F \to \mathbb{G}_a$ is called the logarithm of F, and we denote it by ℓ_F. In particular, we have that

$$\ell_F(F(X,Y)) = \ell(X) + \ell(Y),$$

and

$$F(X,Y) = \ell_F^{-1}(\ell_F(X) + \ell_F(Y)),$$

thus the logarithm series ℓ_F determines the group law F. For formal group laws F and G over a $\mathbb{Q}$-algebra $\mathcal{R}$, the formal power series $\ell_G^{-1}(\ell_F(X)) \in \mathcal{R}[\![X]\!]$ gives an isomorphism $F \to G$ over $\mathcal{R}$.

We will need two formal group laws, D_g and $\hat{E}$, and an isomorphism between them.

The formal group law D_g is associated with the Dirichlet series which has an Euler product. Recall that $g = \sum b(n)q^n \in S_2(N)$ is a primitive form with integral Fourier coefficients $b(n) \in \mathbb{Z}$. It follows from [12] (see also [13], Theorem F) that the formal group $D_g(X,Y) := \ell_{D_g}^{-1}(\ell_{D_g}(X) + \ell_{D_g}(Y))$ determined by its logarithm

$$\ell_{D_g}(q) = \mathcal{E}_g = \sum_{n\geq 1} \frac{b(n)}{n} q^n,$$

which is *a priori* defined over $\mathbb{Q}$, is in fact defined over $\mathbb{Z}$ (that is $D_g(X,Y) \in \mathbb{Z}[\![X,Y]\!]$).

In order to define the formal group law $\hat{E}$ of the elliptic curve (5), let

$$\wp(\Lambda, u) = \wp(u) = \frac{1}{u^2} + \sum_{n\geq 2} c_n u^{2n-2} \in \mathbb{Q}(\!(u)\!),$$

where $c_2 = g_2(\Lambda)/20,\ c_3 = g_3(\Lambda)/28$, etc. be the Laurent series of the Weierstrass $\wp$-function.

We define the formal group law $\hat{E}$ of the elliptic curve (5) as the formal group law determined by its logarithm

$$\ell_{\hat{E}}(X) = \left(-2\frac{\wp}{\wp'}\right)^{-1}(X).$$

The power series

$$f = \ell_{\hat{E}}^{-1}(\ell_{D_g}(q)) = -\frac{2\wp(\mathcal{E}_g)}{\wp'(\mathcal{E}_g)} \in \mathbb{Q}[\![q]\!]$$

gives an isomorphism $D_g \to \hat{E}$ over $\mathbb{Q}$.

The short form of the Weierstrass equation typically is not minimal. One may use the minimal Néron model of E, and produce out of it a formal group law $\tilde{E}$ over $\mathbb{Z}$ using the addition law on E as in Chapter IV.2 of [11]. Then $\tilde{E}$ is isomorphic to our $\hat{E}$ over $\mathcal{Z}$. Honda [12] (see also [13]) proved that the formal group laws $\hat{E}$ and D_g are isomorphic over $\mathbb{Z}$. Our Proposition 10 below is a simplified version of this statement adapted for our purposes.

Proposition 10. *The isomorphism $f : D_g \to \hat{E}$ is defined over $\mathcal{Z}$. In other words,*

$$f = -\frac{2\wp(\mathcal{E}_g)}{\wp'(\mathcal{E}_g)} \in \mathcal{Z}[\![q]\!].$$

Proof. We put $q = e^{2\pi i\tau}$, and consider f as a function of τ which is the pullback of the rational function $Z = -2x/y$ under $\mathcal{E}_g : X_0(N) \to E = \mathbb{C}/\Lambda$. Thus $f(\tau)$ is a meromorphic modular function on $\Gamma_0(N)$. This function is bounded at infinity and has poles in the preimages τ_i of $\frac{1}{2}\Lambda\backslash\Lambda$. Since the map $\mathcal{E}_g$ is an algebraic finite covering map between two algebraic varieties defined over $\overline{\mathbb{Q}}$, the function $f(\tau)$ may have only finitely many poles τ_i with multiplicities κ_i in the fundamental domain $X_0(N) = \overline{\Gamma_0(N)\backslash\mathfrak{H}}$, and the values of the J-invariant $J(\tau_i) \in \overline{\mathbb{Q}}$ at these points are algebraic numbers. The weakly holomorphic weight zero modular form $f(\tau)\prod_i(J(\tau) - J(\tau_i))^{\kappa_i} \in M_0^!(N)$ has algebraic Fourier coefficients and a standard bounded denominators argument based on Theorem 3.52 of [16], implies that $f \in \mathcal{Z}[\![q]\!]$ as required. □

Let F be a formal group law defined over a ring $\mathcal{R}_1$. A ring homomorphism $\mathcal{R}_1 \to \mathcal{R}_2$ allows one to consider F as a formal group law over $\mathcal{R}_2$ (taking the images under the ring homomorphism of all coefficients of the two-variable power series F).

The projection $\mathcal{Z} \twoheadrightarrow \mathbb{Z}/p\mathbb{Z}$ is defined for almost all primes p, and allows us to consider the reductions $F^{(p)}$ over $\mathbb{Z}/p\mathbb{Z}$ of a formal group law F defined over $\mathcal{Z}$. If the characteristic of $\mathcal{R}$ is $p > 0$, then for a formal group law F over $\mathcal{R}$ the largest integer $h = h(F)$ such that the multiplication by p endomorphism $[p]$ is a power series in $\mathcal{R}[\![X^{p^h}]\!]$ is called the height of F. It is known (see e.g. [11]) that $h(F) \geq 1$, and the height is assumed to be infinite when $[p] = 0$. In particular, for all primes $p \nmid N$ (see [12])

$$h(D_g^{(p)}) = \begin{cases} 1 & \text{if } b(p) \neq 0 \\ 2 & \text{if } b(p) = 0. \end{cases}$$

Let us now prove Proposition 4.

Proof. It suffices to show that $\delta(p^{-1}\ell_F(X^p)) \in \mathbb{Z}_p[\![X, Y]\!]$. Let

$$\ell_F(X) = \sum_{n\geq 1} \frac{b(n)}{n} X^n.$$

Then $b(n) \in \mathbb{Z}_p$ by Proposition IV.5.5 of [11]. We have that

$$\delta(\ell_F(X^p)) = \ell_F((F(X,Y)^p) - \ell_F(X^p) - \ell_F(Y^p) = \ell_F((F(X,Y)^p) - \ell_F(F(X^p, Y^p))$$

$$= \sum_{n\geq 1} b(n) \frac{F(X,Y)^{np} - F(X^p, Y^p)^n}{n}.$$

Write $n = mp^\nu$ with $p \nmid m$. We thus need to show that for all $\nu \geq 0$ and $m \geq 1$

$$p^{-\nu}\left(F(X,Y)^{pmp^\nu} - F(X^p, Y^p)^{mp^\nu}\right) \equiv 0 \mod p.$$

Note that

$$F(X,Y)^p \equiv F(X^p, Y^p) \mod p,$$

and write $F(X,Y)^p = F(X^p, Y^p) + pH$ with $H \in \mathbb{Z}_p[\![X, Y]\!]$. It follows from a lemma on binomial coefficients proved by Honda in Lemma 4 in [12], that

$$p^{-\nu}\left((F(X^p, Y^p) + pH)^{mp^\nu} - F(X^p, Y^p)^{mp^\nu}\right) \equiv 0 \mod p,$$

and that is exactly what we need. □

The addition law for the Weierstrass ζ-function

In this section, we prove (9). We write $\wp(z) = \wp(\Lambda, z)$, $\wp'(z) = \wp(\Lambda, z)$, and $\zeta(z) = \zeta(\Lambda, u)$ suppressing the lattice Λ from our notations because the lattice is assumed to be fixed throughout this section. We will make use of classical addition formulas for Weierstrass functions:

$$\wp(u+v) = -\wp(u) - \wp(v) + \frac{1}{4}\left(\frac{\wp'(u)-\wp'(v)}{\wp(u)-\wp(v)}\right)^2, \tag{21}$$

$$\zeta(u+v) = \zeta(u) + \zeta(v) + \frac{1}{2}\frac{\wp'(u)-\wp'(v)}{\wp(u)-\wp(v)}, \tag{22}$$

$$\frac{\wp'}{\wp}(u+v) = -\frac{\wp'(u)}{\wp(u+v)} - \frac{\wp'(v)-\wp'(u)}{\wp(v)-\wp(u)}\left(1-\frac{\wp(u)}{\wp(u+v)}\right). \tag{23}$$

The latter formula can be easily derived from

$$\begin{vmatrix} \wp(u) & \wp'(u) & 1 \\ \wp(v) & \wp'(v) & 1 \\ \wp(u+v) & -\wp'(u+v) & 1 \end{vmatrix} = 0.$$

We now consider the formal group law $\hat{E}$ determined by Equation 5. Let $Z = -2x/y$, and $W = -2/y$. Equation 5 transforms to

$$W = Z^3 - \frac{1}{4}g_2 Z W^2 - \frac{1}{4}g_3 W^3,$$

and, acting as in Chapter IV.1 of [11], we recursively substitute the right-hand side of this equation for W into itself. Then similarly to the proof of Proposition 1.1 a in [11], we make use of a variant of Hensel's lemma ([11], Lemma 1.2) and produce a formal power series

$$W(Z) = Z^3\left(1+\phi\left(Z^2\right)\right)$$

with $\phi(t) \in \mathbb{Z}[1/2, g_2, g_3]\llbracket t\rrbracket \subset \mathcal{Z}\llbracket t\rrbracket$.

Note that the formal power series $X(z) = -2\wp(z)/\wp'(z) \in z\mathbb{Q}\llbracket z\rrbracket$ has rational coefficients, and recall that the logarithm of $\hat{E}$, by definition, is its formal inverse

$$\ell_{\hat{E}} = \left(-2\frac{\wp}{\wp'}\right)^{-1}.$$

Making use of the notations just introduced we firstly rewrite and analyze addition formula (21).

Proposition 11. *In the notations above we have that*

$$\wp(\ell_{\hat{E}}(Z_1) + \ell_{\hat{E}}(Z_2)) = \frac{A(Z_1,Z_2)}{(Z_1+Z_2)^2} \tag{24}$$

with a formal power series $A(Z_1,Z_2) \in \mathcal{Z}\llbracket Z_1,Z_2\rrbracket$ such that $A(0,Z_2) = A(Z_1,0) = 1$.

Proof. Note that (24) is obvious with a formal power series $A(Z_1,Z_2) \in \mathbb{Q}\llbracket Z_1,Z_2\rrbracket$ satisfying $A(Z_1,Z_2) = A(Z_2,Z_1)$ and $A(0,Z_2) = A(Z_1,0) = 1$, because $\wp(z) = 1/z^2 + \ldots$. We thus only need to prove that in fact $A(Z_1,Z_2) \in \mathcal{Z}\llbracket Z_1,Z_2\rrbracket$.

For $i = 1,2$, let $W_i = W(Z_i)$. We rewrite (21) as

$$\wp(\ell_{\hat{E}}(Z_1) + \ell_{\hat{E}}(Z_2)) = -\frac{Z_1}{W_1} - \frac{Z_2}{W_2} + \left(\frac{W_1-W_2}{Z_1W_2 - Z_2W_1}\right)^2.$$

Put for $i = 1,2$ $\psi_i = 1 + \phi(Z_i^2) \in \mathcal{Z}\llbracket Z_i\rrbracket$ to obtain

$$\wp(\ell_{\hat{E}}(Z_1) + \ell_{\hat{E}}(Z_2)) = \frac{1}{Z_1^2Z_2^2}\left(\frac{Z_1^3\psi_1 - Z_2^3\psi_2}{Z_2^2\psi_2 - Z_1^2\psi_1}\right)^2 - \frac{1}{Z_1^2\psi_1} - \frac{1}{Z_2^2\psi_2}.$$

Since

$$\frac{Z_1^3\psi_1 - Z_2^3\psi_2}{Z_1 - Z_2} = \frac{Z_1^3(1+\phi(Z_1^2)) - Z_2^3(1+\phi(Z_2^2))}{Z_1 - Z_2} = Z_1^2 + Z_1Z_2 + Z_2^2 + B(Z_1, Z_2)$$

with $B(Z_1, Z_2) \in Z_1Z_2\mathcal{Z}[\![Z_1, Z_2]\!]$, and

$$\frac{Z_2^2\psi_2 - Z_1^2\psi_1}{Z_2^2 - Z_1^2} = \frac{Z_2^2(1+\phi(Z_2^2)) - Z_1^2(1+\phi(Z_1^2))}{Z_2^2 - Z_1^2} \in 1 + \mathcal{Z}[\![Z_1, Z_2]\!], \tag{25}$$

we obtain

$$\begin{aligned}\wp(\ell_{\hat{E}}(Z_1) + \ell_{\hat{E}}(Z_2)) &= \frac{1}{Z_1^2Z_2^2}\left(\frac{Z_1^2 + Z_1Z_2 + Z_2^2 + B(Z_1,Z_2)}{Z_1 + Z_2}\right)^2 - \frac{1}{Z_1^2\psi_1} - \frac{1}{Z_2^2\psi_2}\\ &= \frac{1}{(Z_1+Z_2)^2}\left(\frac{Z_1}{Z_2} + 1 + \frac{Z_2}{Z_1} + \frac{B(Z_1,Z_2)}{Z_1Z_2}\right)^2 - \frac{1}{Z_1^2\psi_1} - \frac{1}{Z_2^2\psi_2}\\ &= \frac{A(Z_1,Z_2)}{(Z_1+Z_2)^2}\end{aligned}$$

with $A(Z_1, Z_2) \in \mathcal{Z}[\![Z_1, Z_2]\!]$ as required. □

We are now ready to prove (9).

Proposition 12. *Let* $\Phi(Z) = \zeta(\ell_{\hat{E}}(Z)) - 1/Z \in \mathbb{Q}[\![Z]\!]$.
Then $\Phi(\hat{E}(Z_1, Z_2)) - \Phi(Z_1) - \Phi(Z_2) \in \mathcal{Z}[\![Z_1, Z_2]\!]$.

Proof. Note that

$$\hat{E}(Z_1, Z_2) = \ell_{\hat{E}}^{-1}(\ell_{\hat{E}}(Z_1) + \ell_{\hat{E}}(Z_2)) = -\frac{2\wp}{\wp'}(\ell_{\hat{E}}(Z_1) + \ell_{\hat{E}}(Z_2)).$$

For $i = 1, 2$ let $x_i = Z_i/W_i$ and $y_i = -2/W_i$.
Using (22) and (23) we find that

$$\Phi(\hat{E}(X_1, X_2)) - \Phi(X_1) - \Phi(X_2) =$$
$$\frac{1}{2}\frac{y_1 - y_2}{x_1 - x_2} - \frac{1}{2}\left(\frac{y_2 - y_1}{x_2 - x_1}\left(1 - \frac{x_1}{\wp(\ell_{\hat{E}}(Z_1) + \ell_{\hat{E}}(Z_2))}\right) + \frac{y_1}{\wp(\ell_{\hat{E}}(Z_1) + \ell_{\hat{E}}(Z_2))}\right) + \frac{1}{Z_1} + \frac{1}{Z_2}.$$

We take into the account that

$$x_1\frac{y_2 - y_1}{x_2 - x_1} - y_1 = 2\frac{Z_2 - Z_1}{Z_2W_1 - Z_1W_2} = \frac{1}{Z_1Z_2}\frac{Z_2 - Z_1}{Z_1^2\psi_1 - Z_2^2\psi_2},$$

and make use of (24) to find that

$$\Phi(\hat{E}(Z_1, Z_2)) - \Phi(Z_1) - \Phi(Z_2) = \frac{(Z_1 + Z_2)^2}{A(Z_1, Z_2)}\frac{1}{Z_1Z_2}\frac{Z_2 - Z_1}{Z_1^2\psi_1 - Z_2^2\psi_2} + \frac{1}{Z_1} + \frac{1}{Z_2}$$

with $\psi_i = 1 + \phi(Z_i^2) \in \mathcal{Z}[\![Z_i]\!]$ as above. Taking into the account (25) and Proposition 11, we conclude that

$$\Phi(\hat{E}(Z_1, Z_2)) - \Phi(Z_1) - \Phi(Z_2) \in \left(\frac{1}{Z_1} + \frac{1}{Z_2}\right)\mathcal{Z}[\![Z_1, Z_2]\!].$$

At the same time, since $\Phi(Z) = \zeta(\ell_{\hat{E}}(Z)) - 1/Z \in \mathbb{Q}[\![Z]\!]$, clearly

$$\Phi(\hat{E}(Z_1, Z_2)) - \Phi(Z_1) - \Phi(Z_2) \in \mathbb{Q}[\![X]\!].$$

We thus conclude that $\Phi(\hat{E}(Z_1, Z_2)) - \Phi(Z_1) - \Phi(Z_2) \in \mathcal{Z}[\![Z_1, Z_2]\!]$ as required. □

Conclusion

We want to emphasize the similarity between (7) and the decomposition

$$\zeta(\Lambda, \mathcal{E}_g) + \frac{\wp'(\Lambda, \mathcal{E}_g)}{2\wp(\Lambda, \mathcal{E}_g)} \equiv \lambda_p \mathcal{E}_g + \mu_p Fr(\mathcal{E}_g) \mod \mathbb{Z}_p[\![q]\!]$$

derived from (10) and (8) valid for all but finitely many primes p. Here $Fr(\mathcal{E}_g)(q) = \mathcal{E}_g(q^p)$ is the action of Frobenius on H^1_{dR} of the formal group law over $\mathbb{Z}_p$.

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References

1. Miyake, T: Modular forms. Translated from the 1976 Japanese original by Yoshitaka Maeda. Reprint of the first 1989 English edition. Springer Monographs in Mathematics. Springer-Verlag, Berlin (2006)
2. Bruinier, JH, Ono, K, Rhoades, R: Differential operators for harmonic weak Maass forms and the vanishing of Hecke eigenvalues. Math. Ann. **342**, 673–693 (2008)
3. Guerzhoy, P, Kent, ZA, Ono, K: *p*-adic coupling of mock modular forms and shadows. Proc. Natl. Acad. Sci. USA. **107**(14), 6169–6174 (2010)
4. Bringmann, K, Guerzhoy, P, Kane, B: Mock modular forms as *p*-adic modular forms. Trans. Amer. Math. Soc. **364**, 2393–2410 (2012)
5. Elkies, ND: The existence of infinitely many supersingular primes for every elliptic curve over Q. Invent. Math. **89**(3), 561–567 (1987)
6. Zagier, D: Modular parametrizations of elliptic curves. Canad. Math. Bull. **28**(3), 372–384 (1985)
7. Mazur, B, Swinnerton-Dyer, P: Arithmetic of Weil curves. Invent. Math. **25**, 1-61 (1974)
8. Weil, A: Elliptic functions according to Eisenstein and Kronecker. Reprint of the 1976 original. Classics in Mathematics. Springer-Verlag, Berlin (1999)
9. Katz, NM: p-adic interpolation of real analytic Eisenstein series. Ann. Math. **104**(3), 459–571 (1976)
10. Katz, NM: Crystalline cohomology, Dieudonné modules, and Jacobi sums. Automorphic forms, representation theory and arithmetic (Bombay, 1979), pp. 165–246. Tata Inst. Fund. Res. Studies in Math., 10, Tata Inst. Fundamental Res., Bombay (1981)
11. Silverman, JH: The arithmetic of elliptic curves. Corrected reprint of the 1986 original. Graduate Texts in Mathematics, 106. Springer-Verlag, New York (1992)
12. Honda, T: Formal groups and zeta-functions. Osaka J. Math. **5**, 199–213 (1968)
13. Hill, WL: Formal groups and zeta-functions of elliptic curves. Invent. Math. **12**, 321–336 (1971)
14. Ono, K: Unearthing the visions of a master: harmonic Maass forms and number theory, Proceedings of the 2008 Harvard-MIT Current Developments in Mathematics Conference, pp 347–454. International Press, Somerville (2009)
15. Lang, S: Introduction to modular forms. With appendixes by D. Zagier and Walter Feit. Corrected reprint of the 1976 original. Grundlehren der Mathematischen Wissenschaften, 222. Springer-Verlag, Berlin (1995)
16. Shimura, G: Introduction to the arithmetic theory of automorphic functions. Reprint of the 1971 original. Publications of the Mathematical Society of Japan, 11. Kanô Memorial Lectures, 1. Princeton University Press, Princeton (1994)

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